

# Besov regularity of stochastic measures

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## Abstract

We prove that continuous paths of  $\sigma$ -additive in probability set function belong to Besov space.

The Besov regularity of trajectories has been studied for some special classes of stochastic processes. Brownian motion trajectories have been studied in Roynette (1993). Other Gaussian processes were considered in Ciesielski, Kerkycharian and Roynette (1993). The Besov regularity of indefinite Skorohod integral w.r.t. fractional Brownian motion was studied in Lakhel, Ouknine and Tudor (2002), Nualart and Ouknine (2003).

In the given note we consider the class of continuous stochastic processes generated by values of stochastic measures and prove the Besov regularity of their paths.

Let  $L_0 = L_0(\Omega, \mathcal{F}, \mathbf{P})$  be a set of all real-valued random variables defined on the probability space  $(\Omega, \mathcal{F}, \mathbf{P})$  (more precisely, the set of equivalence classes). Convergence in  $L_0$  means the convergence in probability. Let  $\mathbf{X}$  be an arbitrary set and  $\mathcal{B}$  be a  $\sigma$ -algebra of subsets of  $\mathbf{X}$ .

**Definition** *The  $\sigma$ -additive mapping  $\mu : \mathcal{B} \rightarrow L_0$  is called a stochastic measure.*

In other words,  $\mu$  is a vector measure with values in  $L_0$ . We do not assume positivity or moment existence for  $\mu$ . In Kwapień and Woyczyński (1992) such  $\mu$  is called a general stochastic measure.

Examples of stochastic measures are the following.

Let  $\mathbf{X} = [a, b] \subset \mathbb{R}_+$ ,  $\mathcal{B}$  be the  $\sigma$ -algebra of Borel subsets of  $[a, b]$ , and  $W(t)$  be the Brownian motion. Then  $\mu(A) = \int_a^b I_A(t) dW(t)$  is a stochastic measure on  $\mathcal{B}$ . Moreover by the same way any continuous square integrable martingale  $X(t)$ ,  $a \leq t \leq b$  defines a stochastic measure  $\mu$  on  $\mathcal{B}$  so that  $\mu((s, t]) = X(t) - X(s)$ . If  $W^H(t)$  is a fractional Brownian motion with Hurst index  $H > 1/2$  and  $f : [0, T] \rightarrow \mathbb{R}$  is a bounded measurable function then  $\mu(A) = \int_0^T f(t) I_A(t) dW^H(t)$  is a stochastic measure on  $\mathcal{B}$  too (this fact follows from Theorem 1.1 of Memin, Mishura and Valkeila, 2001). Other examples may be found in subsection 7.2 of Kwapień and Woyczyński (1992).

Let us consider an arbitrary stochastic process  $X(t)$ ,  $a \leq t \leq b$ . Put  $\mu((s, t]) = X(t) - X(s)$  and extend  $\mu$  to algebra  $\mathcal{B}_0$  of all finite unions  $\cup_{k=1}^l (a_k, b_k] \subset (a, b]$  by additivity. Then  $\mu$  can be extended to a stochastic measure on the Borel  $\sigma$ -algebra iff both the following conditions holds:

(i)  $\mu(A_n) \xrightarrow{\mathbf{P}} 0$  for any  $A_n \in \mathcal{B}_0$ ,  $A_n \downarrow \emptyset$ ,

(ii) the set of random variables  $\{\mu(A_n), n \geq 1\}$  is bounded in probability for any disjoint  $A_n \in \mathcal{B}_0$

(Theorem 1 of Radchenko, 1991).

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It is known that the for any  $\mathbf{X}$ ,  $\mathcal{B}$  the set of values of any stochastic measure is bounded in probability, i. e.

$$\lim_{c \rightarrow \infty} \sup_{A \in \mathcal{B}} \mathbf{P}(|\mu(A)| > c) = 0$$

(see Talagrand, 1981). Furthermore Theorem 7.1.2 of Kwapień and Woyczyński (1992) establishes that the set

$$\left\{ \sum_{k=1}^n c_k \mu(A_k), A_{k_1} \cap A_{k_2} = \emptyset \text{ for } k_1 \neq k_2, n \geq 1, |c_k| \leq 1 \right\} \quad (1)$$

is bounded in probability.

We consider the *Besov space*  $B_{pq}^\alpha([a, b])$ ,  $[a, b] \subset \mathbb{R}$ . Recall that the norm in this classical space for  $1 \leq p, q < \infty$  and  $0 < \alpha < 1$  may be introduced by

$$\|f\|_{p,q}^\alpha = \|f\|_{L_p([a, b])} + \left( \int_0^{b-a} (w_p(t, f))^q t^{-\alpha q - 1} dt \right)^{1/q},$$

where

$$w_p(t, f) = \sup_{|h| \leq t} \left( \int_{I_h} |f(x-h) - f(x)|^p dx \right)^{1/p}, \quad I_h = \{x \in [a, b] : x-h \in [a, b]\}.$$

The norm in the Besov space  $B_{pp}^\alpha([a, b])$  is equivalent to the norm in the *Slobodeckij space*  $W_p^\alpha$ .

The main result of the paper will be based on Corollary 3.3 of Kamont (1997) and the following property of stochastic measures.

**Lemma** *Let  $\mu$  be a stochastic measure and  $a_n$ ,  $n \geq 1$ , be a sequence of positive numbers such that  $\sum_{n=1}^\infty a_n < \infty$ . Let  $\Delta_{kn} \in \mathcal{B}$ ,  $n \geq 1$ ,  $1 \leq k \leq l_n$ , be such that for each  $n$  and  $k_1 \neq k_2$   $\Delta_{k_1 n} \cap \Delta_{k_2 n} = \emptyset$ . Then*

$$\sum_{n=1}^\infty a_n^2 \sum_{k=1}^{l_n} \mu^2(\Delta_{kn}) < \infty \quad a. s.$$

**Proof.** Suppose that

$$\mathbf{P} \left[ \sum_{n=1}^\infty a_n^2 \sum_{k=1}^{l_n} \mu^2(\Delta_{kn}) = +\infty \right] = \varepsilon_0 > 0.$$

We find that for any  $c > 0$  there exists  $j$  such that

$$\mathbf{P} \left[ \sum_{n=1}^j a_n^2 \sum_{k=1}^{l_n} \mu^2(\Delta_{kn}) \geq c \right] \geq \varepsilon_0/2. \quad (2)$$

For  $j$  from (2), let us consider the set

$$\Omega_1 = \left\{ \omega \in \Omega : \sum_{n=1}^j a_n^2 \sum_{k=1}^{l_n} \mu^2(\Delta_{kn}) \geq c \right\},$$

and the set of independent symmetric Bernoulli random variables  $\varepsilon_{kn}$ ,  $1 \leq n \leq j$ ,  $1 \leq k \leq l_n$ , defined on other probability space  $(\Omega', \mathcal{F}', \mathbf{P}')$ ,  $\mathbf{P}'(\varepsilon_{kn} = 1) = \mathbf{P}'(\varepsilon_{kn} = -1) = 1/2$ . We have the following consequence of Paley-Zigmund inequality

$$\mathbf{P}' \left[ \left( \sum_{1 \leq n \leq j, 1 \leq k \leq l_n} \lambda_{kn} \varepsilon_{kn} \right)^2 \geq (1/4) \sum_{1 \leq n \leq j, 1 \leq k \leq l_n} \lambda_{kn}^2 \right] \geq 1/8, \quad \lambda_{kn} \in \mathbb{R}$$

(see, for example, Lemma V.4.3 (a) Vakhania, Tarieladze and Chobanian (1987) or Lemma 0.2.1 Kwapien and Woyczyński (1992) for  $\lambda = 1/4$ ).

By applying this inequality with taking  $\lambda_{kn} = a_n \mu(\Delta_{kn}, \omega)$  for a fixed  $\omega \in \Omega_1$  we get

$$P' \left[ \omega' : \left( \sum_{n=1}^j a_n \sum_{k=1}^{l_n} \varepsilon_{kn}(\omega') \mu(\Delta_{kn}, \omega) \right)^2 \geq c/4 \right] \geq 1/8.$$

Integrating the above inequality with respect to measure  $P$  over  $\Omega_1$  we obtain

$$P \times P' \left[ (\omega, \omega') : \left( \sum_{n=1}^j a_n \sum_{k=1}^{l_n} \varepsilon_{kn}(\omega') \mu(\Delta_{kn}, \omega) \right)^2 \geq c/4 \right] \geq \varepsilon_0/16.$$

Hence by using Fubini's theorem, we have that there exists  $\omega'_0 \in \Omega'$  such that

$$P \left[ \omega : \left( \sum_{n=1}^j a_n \sum_{k=1}^{l_n} \varepsilon_{kn}(\omega'_0) \mu(\Delta_{kn}, \omega) \right)^2 \geq c/4 \right] \geq \varepsilon_0/16.$$

Recalling that each  $\varepsilon_{kn}(\omega'_0) = 1$  or  $\varepsilon_{kn}(\omega'_0) = -1$  we obtain sets  $B_n, C_n \in \mathcal{B}$  such that

$$P \left[ \left| \sum_{n=1}^j a_n (\mu(B_n) - \mu(C_n)) \right| \geq \sqrt{c}/2 \right] \geq \varepsilon_0/16. \quad (3)$$

We have

$$\max_{x \in X} \left| \sum_{n=1}^j a_n (1_{B_n}(x) - 1_{C_n}(x)) \right| \leq \sum_{n=1}^{\infty} a_n. \quad (4)$$

Recall that  $\varepsilon_0 > 0$  is fixed and  $c > 0$  is arbitrary. Therefore (3) and (4) contradict the boundedness in probability of the sums (1). This completes the proof of the Lemma.  $\square$

The main result of the paper is the following.

**Theorem** Let  $X = [a, b] \subset \mathbb{R}$ ,  $\mathcal{B}$  be the Borel  $\sigma$ -algebra,  $\mu$  be a stochastic measure on  $\mathcal{B}$  and the process  $\mu(t) = \mu([a, t])$ ,  $a \leq t \leq b$ , have continuous paths. Then for any  $p \geq 2$ ,  $0 < \alpha < 1/p$ , the path of  $\mu(t)$  with probability 1 belongs to the Besov space  $B_{pp}^\alpha([a, b])$ .

**Proof.** When  $f : [a, b] \rightarrow \mathbb{R}$  is a continuous function, Corollary 3.3 of Kamont (1997) shows that the convergence of the series

$$\sum_{n=1}^{\infty} 2^{n(\alpha p - 1)} \sum_{k=1}^{2^n} |f(a + k2^{-n}(b-a)) - f(a + (k-1)2^{-n}(b-a))|^p$$

implies that  $f \in B_{pp}^\alpha([a, b])$ . Obviously, it is sufficient to prove the convergence for the second power of differences of function values.

By taking

$$a_n = 2^{n(\alpha p - 1)/2}, \quad \Delta_{kn} = (a + (k-1)2^{-n}(b-a), a + k2^{-n}(b-a)], \quad 1 \leq k \leq 2^n$$

in the Lemma, we see that continuous paths of  $\mu(t)$  a. s. satisfy the mentioned condition.  $\square$

In particular the statement of the Theorem may be applied to the paths of any continuous square integrable martingale.

Embeddings of the Besov spaces (see, for example, subsection 3.2.4 of Triebel, 1983) implies that a continuous path of  $\mu(t)$  a. s. belongs to  $B_{pq}^\alpha([a, b])$ ,  $q \geq p \geq 2$ ,  $0 < \alpha < 1/p$ .

Note that with probability 1 the trajectories of the Brownian motion do not belong to the Besov spaces  $B_{pq}^\alpha([0, 1])$  for all  $1/2 < \alpha < 1$ ,  $p, q \geq 1$  (Theorem 1 of Roynette, 1993).

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